

ANNULAR-EFFICIENT TRIANGULATIONS OF 3-MANIFOLDS

WILLIAM JACO AND J. HYAM RUBINSTEIN

ABSTRACT. A triangulation of a compact 3-manifold is annular-efficient if it is 0-efficient and the only normal, incompressible annuli are thin edge-linking. If a compact 3-manifold has an annular-efficient triangulation, then it is irreducible, ∂ -irreducible, and an-annular. Conversely, it is shown that for a compact, irreducible, ∂ -irreducible, and an-annular 3-manifold, any triangulation can be modified to an annular-efficient triangulation. It follows that for a manifold satisfying this hypothesis, there are only a finite number of boundary slopes for incompressible and ∂ -incompressible surfaces of a bounded Euler characteristic.

1. INTRODUCTION

In this paper we connect interesting properties of ideal triangulations of the interiors of compact 3-manifolds with interesting properties of triangulations of the compact 3-manifold by exploiting the inverse relationship between crushing a triangulation along a normal surface [5] and that of inflating an ideal triangulation [9]. In [5] it is shown that a compact, irreducible, ∂ -irreducible, and an-annular 3-manifold admits an ideal triangulation of its interior. Here we show that any triangulation of such a 3-manifold can be modified to an ideal triangulation of the interior of the manifold; hence, providing a construction for such ideal triangulations. In [5] it also is shown that one can get 0-efficient ideal triangulations of the interiors of these manifolds. Here we show that we actually can construct ideal triangulations that satisfy a stronger condition that implies 0-efficient.

Marc Lackenby proves in [13] that under our hypothesis and with the additional condition that each boundary component of the manifold is a torus, then these manifolds admit a taut ideal triangulation of their interiors. We were not able to show our construction will give taut ideal triangulations in the case of tori boundaries; indeed, at this point and in the case of tori boundaries, tautness appears to be a stronger condition on an ideal triangulation than those we have. It is shown in [11] that if in addition the manifold is atoroidal, then a taut ideal triangulation is not only 0-efficient but is also 1-efficient in the sense that the only normal tori are vertex-linking.

Date: August 16, 2011.

1991 *Mathematics Subject Classification.* Primary 57N10, 57M99; Secondary 57M50.

Key words and phrases. ideal triangulation, one-vertex triangulation, layered triangulation, inflation, normal surface, vertex-linking, crushing, frame, slope, Dehn-filling, exceptional surgery.

The first author was partially supported by NSF/DMS Grants, The Grayce B. Kerr Foundation, The American Institute of Mathematics (AIM), and and The Visiting Research Scholar Program at University of Melbourne (Australia).

The second author was partially supported by The Australian Research Council and The Grayce B. Kerr Foundation.

In Section 2, we define what we mean for a triangulation of a compact 3-manifold with boundary to have a normal boundary and the notion of a normal surface being normally isotopic into the boundary. Theorem 2.1 establishes for any triangulation of the manifolds we are interested in, there is an algorithm to decide if there is a closed normal surface that is isotopic into the boundary but is not normally isotopic into the boundary. Furthermore, if there is one the algorithm will construct one. In [5] it is shown that given any triangulation, it can be decided if the triangulation is 0-efficient. In Proposition 4.2 we show the analogous result that it can be decided if a triangulation is annular-efficient. Furthermore, if it is not annular-efficient, the algorithm constructs a non vertex-linking normal 2-sphere or disk, or if there are none of these, it then constructs a non thin edge-linking normal annulus. A triangulation having the property that any normal surface isotopic into the boundary must be normally isotopic into the boundary can be regarded as a ∂ -efficient condition in the sense we use 0-efficient, annular-efficient, and 1-efficient.

In Section 3 we review crushing triangulations along normal surfaces and inflating ideal triangulations. An inflation is defined in terms of a combinatorial crushing; this enables us to establish very strong relationships between ideal triangulations and their inflations. In particular, in Theorem 3.5 we establish a one-one correspondence between the closed normal surfaces in an ideal triangulation and the closed normal surface in any of its inflations; furthermore, we show corresponding surfaces under this correspondence are homeomorphic.

In Section 4 we have our main result:

Theorem 4.5. *Suppose $M \neq \mathbb{B}^3$ is a compact, irreducible, ∂ -irreducible, an-annular 3-manifold with nonempty boundary. Then there is an algorithm that will modify any triangulation of M to an annular-efficient triangulation of M .*

We use annular-efficient triangulations to show that for a compact, irreducible, ∂ -irreducible, and an-annular 3-manifold there are only a finite number of boundary slopes possible for incompressible and ∂ -incompressible surfaces having a bounded Euler characteristic. It has been communicated to us by David Bachman and Saul Schleimer that they have independently obtained a similar result.

2. TRIANGULATIONS AND NORMAL SURFACES

We continue with the use of (pseudo) triangulations and ideal triangulations as in [5].

If $\tilde{\Delta}$ is a pairwise disjoint collection of oriented tetrahedra and Φ is a family of orientation-reversing face identifications of the tetrahedra in $\tilde{\Delta}$, then the identification space $X = \tilde{\Delta}/\Phi$ is a 3-complex and is a 3-manifold at each point except possibly at the vertices. If X is a manifold, we denote the collection of tetrahedra and the face identifications by a single symbol \mathcal{T} and say \mathcal{T} is a *triangulation* of the manifold X . If X is not a manifold, then $X \setminus \{\text{vertices}\}$ is the interior of a compact 3-manifold with boundary, M , and we say \mathcal{T} is an *ideal triangulation* of $\overset{\circ}{M}$, the interior of M ; in this case we also say that X is a *pseudo-manifold* and \mathcal{T} is an ideal triangulation of X . For an ideal triangulation, the image of a vertex of a tetrahedron in $\tilde{\Delta}$ is called an *ideal vertex* and its *index* is the genus of its vertex-linking surface. For an ideal triangulation we will always assume the index of each vertex is ≥ 1 .

For our triangulations, the simplices of $\tilde{\Delta}$ are not necessarily embedded in X ; however, the interior of each simplex is embedded. We call the image in X of a tetrahedron, face, or edge in $\tilde{\Delta}$, a tetrahedron, face, or edge. For a tetrahedron Δ in X , there is precisely one tetrahedron $\tilde{\Delta}$ in $\tilde{\Delta}$ that projects to Δ , called the *lift of Δ* . For a face σ in X , there are either one or two faces in $\tilde{\Delta}$ that project to σ ; if only one face projects to σ , then σ is in the boundary of M . If e is an edge in X , the number of edges in $\tilde{\Delta}$ that project to e is the *index of e* .

See [5] for more details regarding triangulations from our point of view.

2.1. Normal surfaces. If M is a 3-manifold and \mathcal{T} is a triangulation of M , we say the properly embedded surface S in M is *normal* (with respect to \mathcal{T}) if for every tetrahedron Δ in $\tilde{\Delta}/\Phi$, the intersection of S with Δ lifts to a collection of normal triangles and normal quadrilaterals in $\tilde{\Delta}$, the lift of Δ . Note that since our tetrahedra have possible face identifications, the intersection of a normal surface with a tetrahedron need not be a normal triangle or a normal quadrilateral but might be one of these with edge identifications.

We shall assume the reader is familiar with classical normal surface theory, which carries over in all of our situations.

A triangulation of a compact 3-manifold with boundary is said to be a *normal boundary triangulation* or to have a *normal boundary* if the frontier of a small regular neighborhood of the boundary is normally isotopic to a normal surface. In this case, we call the normal surface consisting of the frontier of a small regular neighborhood of the boundary the *normal boundary*. Not all triangulations have a normal boundary; for example, layered triangulations of handlebodies [10] contain no closed normal surfaces and, hence, can not have a normal boundary.

A properly embedded surface in a compact 3-manifold with boundary is said to be *isotopic into ∂M* if there is an isotopy of the surface through M into ∂M keeping the boundary of the surface fixed. If the manifold is triangulated and the surface is closed and normal, it is said to be *normally isotopic into ∂M* , if the triangulation has normal boundary and the surface is normally isotopic to the normal boundary. We are interested in triangulations in which the only closed, normal surface isotopic into the boundary is the normal boundary.

A properly embedded annulus in a 3-manifold is *essential* if it is incompressible and not isotopic into the boundary. A compact 3-manifold is said to be *an-annular* if it has no properly embedded, essential annuli.

The following theorem gives conditions under which we can decide if a triangulation of a compact 3-manifold with boundary has a closed, normal surface that is isotopic into the boundary but is not normally isotopic into the boundary.

2.1. Theorem. *Suppose M is a compact, orientable 3-manifold with boundary that is irreducible, ∂ -irreducible, and an-annular. Then for any triangulation \mathcal{T} of M there is an algorithm to decide if there is a closed normal surface that is isotopic into ∂M but is not normally isotopic into ∂M . Furthermore, if there is one the algorithm will construct one.*

Proof. If S and S' are disjoint normal surfaces embedded in M and both are isotopic into ∂M , we say S' is *larger than S* if S is contained in the product region between S' and ∂M . Being larger than is a partial order on closed normal surfaces embedded in M .

Suppose there is a normal surface in M that is isotopic into ∂M but is not normally isotopic into ∂M . By Kneser's Finiteness Theorem [12] there are maximal (relative to the preceding partial order) such surfaces. Suppose S is a maximal normal surface that is isotopic into ∂M but not normally isotopic into ∂M . We claim S is a fundamental surface.

Suppose S is not fundamental. Then $S = X + Y$ is a nontrivial Haken sum. Hence, there are exchange annuli between X and Y . Suppose A is an exchange annulus. Then A is a 0-weight annulus meeting S only in its boundary. There are two possibilities: either A is not in the product region between S and ∂M or A is in the product region between S and ∂M . Since S is isotopic into ∂M and M is an-annular, then for either possibility, A is isotopic into S .

Let $N = N(S \cup A)$ be a small regular neighborhood of $S \cup A$, then N has three boundary components; one is a torus bounding a solid torus, which is a product between A and an annulus A' in S , another is surface normally isotopic to S , and the third is a surface isotopic to S but possibly not normal and even if normal is not normally isotopic to S . The complex $S \cup A$ is a barrier (see [5]) and thus each boundary component of N can be normalized in the closure of the component of its complement not meeting $S \cup A$.

Suppose A is not in the closure of the product region between S and ∂M . Then the component of ∂N isotopic to S can be normalized missing $S \cup A$ to a normal surface S' . Since M is irreducible and ∂ -irreducible, S' is isotopic to S and therefore, isotopic into ∂M . Moreover S is not normally isotopic to S or into ∂M due to the annulus A . But S' is larger than S , which contradicts S being maximal.

Suppose A is in the closure of the product region between S and ∂M . Then A co-bounds a solid torus which is a product between A and an annulus A' in S . We observe that $X \neq Y$, for if this were not the case, then X (and Y) would be one-sided and M would be a twisted I-bundle, contradicting M being an-annular. Hence, there must be a trace curve in A' . Suppose we have selected A' in this situation so that it has a minimal number of trace curves. Since there is a trace curve in A' , there is another exchange annulus A_1 for S meeting A' in at least one of its boundary components. If A_1 is not in the closure of the product region between S and ∂M , then the preceding argument gives a contradiction to our selection of S . So we may assume A_1 is, like A , in the closure of the product region between S and ∂M and therefore in the solid torus co-bounded by A and A' . It follows that A_1 co-bounds a solid torus which is a product between A_1 and an annulus A'_1 in $A' \subset S$. However, then A'_1 has fewer trace curves than A' contradicting our choice of the exchange annulus A .

So, there is a closed normal surface that is isotopic into ∂M and not normally isotopic into ∂M if and only if there is such a surface among the fundamental surfaces for the triangulation \mathcal{T} of M . By [8] given any normal surface we can determine if it is isotopic into ∂M and it is straight forward to recognize if it is normally isotopic into ∂M . It follows if such a surface exists, we can construct one. \square

The argument carries over to an analogous result in the case of an ideal triangulation.

2.2. Corollary. *Suppose M is a compact, orientable 3-manifold with boundary that is irreducible, ∂ -irreducible, and an-annular. Then for any ideal triangulation \mathcal{T}^**

of $\overset{\circ}{M}$ and any ideal vertex v^* of \mathcal{T}^* , there is an algorithm to decide if there is a closed normal surface that is isotopic into the vertex-linking surface of v^* but is not normally isotopic into the vertex-linking surface of v^* . Furthermore, if there is one, the algorithm will construct one.

3. BASICS OF CRUSHING AND INFLATING TRIANGULATIONS

3.1. Crushing triangulations along normal surfaces. In [5] we introduced the procedure of “crushing a triangulation along a normal surface.” Details may be reviewed there, as well as in [9], where the details apply more directly to our situation in this work.

Suppose \mathcal{T} is a triangulation of the compact 3-manifold M or an ideal triangulation of the interior of M . Suppose S is a closed normal surface in M , X is the closure of a component of the complement of S , and X does not contain any of the vertices of \mathcal{T} . Since X does not contain any of the vertices of \mathcal{T} , the triangulation \mathcal{T} induces a particularly nice cell-decomposition on X , say \mathcal{C}_X , consisting of *truncated-tetrahedra*, *truncated-prisms*, *triangular product blocks*, and *quadrilateral product blocks*. See Figure 1.

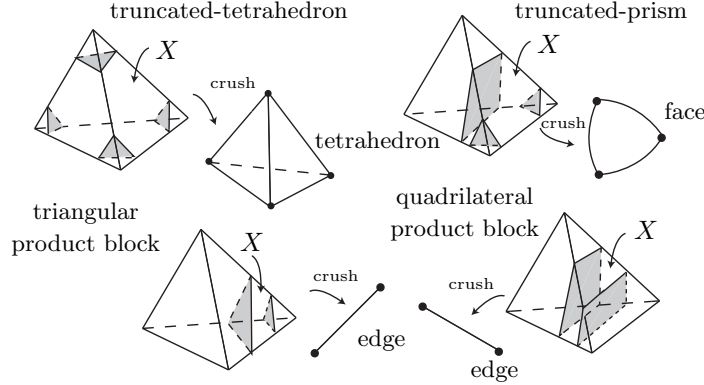


FIGURE 1. Cells in induced cell-decomposition \mathcal{C}_X of X and their crushing to tetrahedra, faces, and edges in an ideal triangulation of $\overset{\circ}{X}$.

The boundary of each 3-cell in \mathcal{C}_X has an induced cell decomposition in which some of the cells are in S and some are not. The edges and faces in the decomposition \mathcal{C}_X are called *horizontal* if their interiors are in S and *vertical* if their interiors are not in S . The quadrilateral vertical 2-cells are called *trapezoids*; there are two in a truncated-prism, three in a triangular block, and four in a quadrilateral block. The non-trapezoidal vertical 2-cells are in truncated-prisms and truncated-tetrahedra and are hexagons.

We define $\mathbb{P}(\mathcal{C}_X)$ as the union, $\mathbb{P}(\mathcal{C}_X) = \{\text{vertical edges of } \mathcal{C}_X\} \cup \{\text{trapezoids}\} \cup \{\text{triangular blocks}\} \cup \{\text{quadrilateral blocks}\}$. Each component of $\mathbb{P}(\mathcal{C}_X)$ is an I -bundle. Suppose each component of $\mathbb{P}(\mathcal{C}_X)$ is a product I bundle. Then a component of $\mathbb{P}(\mathcal{C}_X)$ is a product $\mathbb{P}_i = K_i \times I$, where $K_i^\varepsilon = K_i \times \varepsilon$, $\varepsilon = 0, 1$ and $K_i \times 0$ and $K_i \times 1$ are isomorphic subcomplexes in the induced normal cell decomposition on S , $i = 1, 2, \dots, k$, k being the number of components of $\mathbb{P}(\mathcal{C}_X)$. In this situation,

we call $\mathbb{P}(\mathcal{C}_X)$ the *combinatorial product for \mathcal{C}_X* . If $\mathbb{P}(\mathcal{C}_X) \neq X$ and each K_i is a simply connected planar complex (hence, it is cell-like), we say $\mathbb{P}(\mathcal{C}_X)$ is a *trivial combinatorial product*. In applications, we do not always have things so nice and we need to modify $\mathbb{P}(\mathcal{C}_X)$ to an *induced product region for X* , denoted $\mathbb{P}(X)$.

Now, consider the truncated-prisms in \mathcal{C}_X . Each truncated-prism has two hexagonal faces. In \mathcal{C}_X , these hexagonal faces are identified via the face identifications of the given triangulation \mathcal{T} to a hexagonal face of a truncated-tetrahedron or to a hexagonal face of truncated-prism. If we follow a sequence of such identifications through hexagonal faces of truncated-prisms, we trace out a well-defined arc that terminates at an identification with a hexagonal face of a truncated-tetrahedron or possibly does not terminate but forms a complete cycle through hexagonal faces of truncated-prisms. We call a collection of truncated-prisms identified in this way a *chain*. If a chain ends in a truncated-tetrahedra, we say the chain *terminates*; otherwise, we call the chain a *cycle of truncated-prisms*.

Just as in [9], under appropriate conditions, we can construct an ideal triangulation of $\overset{\circ}{X}$ using a controlled crushing of the cells of \mathcal{C}_X . In particular, to obtain the desired ideal triangulation of $\overset{\circ}{X}$ it is sufficient that $X \neq \mathbb{P}(\mathcal{C}_X)$ or in the more general case $X \neq \mathbb{P}(X)$ (there are not too many product blocks) and there are no cycles of truncated-prisms (there are not too many truncated-prisms). As a result of the crushing, each component of S is crushed to a point (distinct points for distinct components), all designated products are crushed to arcs and, in particular, the products $K_i \times I$ are crushed to arcs (edges) so that if $K_i \times I$ is crushed to the edge e_i , then the crushing projection coincides with the projection of $K_i \times I$ onto the I factor. Vertical edges, trapezoids, and product blocks in \mathcal{C}_X are identified to edges in the ideal triangulation. Truncated-prisms becomes faces and truncated-tetrahedra become tetrahedra. Consult [5] and see Figure 1.

The crushing is particularly nice in the case that $\mathbb{P}(\mathcal{C}_X)$ is a trivial combinatorial product, $X \neq \mathbb{P}(\mathcal{C}_X)$, and there are no cycles of truncated prisms. In this case, suppose $\{\overline{\Delta}_1, \dots, \overline{\Delta}_n\}$ denotes the collection of truncated-tetrahedra in \mathcal{C}_X . Each truncated-tetrahedron in \mathcal{C}_X has its triangular faces in S . If we crush each such triangular face of a truncated-tetrahedron to a point (for the moment, distinct points for each triangular face), we get a tetrahedron. We use the notation $\tilde{\Delta}_i^*$ for the tetrahedron coming from the truncated-tetrahedron $\overline{\Delta}_i$ after identifying the triangular faces of $\overline{\Delta}_i$ to points. Also as a consequence of this crushing of S , if $\overline{\sigma}_i$ is a hexagonal face in $\overline{\Delta}_i$, then $\overline{\sigma}_i$ is identified to a triangular face, say $\tilde{\sigma}_i^*$, of $\tilde{\Delta}_i^*$.

Let $\tilde{\Delta}^* = \{\tilde{\Delta}_1^*, \dots, \tilde{\Delta}_n^*\}$ be the tetrahedra obtained from the collection of truncated-tetrahedra $\{\overline{\Delta}_1, \dots, \overline{\Delta}_n\}$ following the crushing of the normal triangles in the surface S to points. It follows that there is a family Φ^* of face-pairings induced on the collection of tetrahedra $\tilde{\Delta}^*$ by the face-pairings of \mathcal{C}_X (coming from the face-pairings of \mathcal{T}) as follows:

- if the face $\overline{\sigma}_i$ of $\overline{\Delta}_i$ is paired with the face $\overline{\sigma}_j$ of $\overline{\Delta}_j$, then this pairing induces the pairing of the face $\tilde{\sigma}_i^*$ of $\tilde{\Delta}_i^*$ with the face $\tilde{\sigma}_j^*$ of $\tilde{\Delta}_j^*$;
- if the face $\overline{\sigma}_i$ of $\overline{\Delta}_i$ is paired with a face of a truncated-prism in a chain of truncated-prisms and the face $\overline{\sigma}_j$ of the truncated-tetrahedron $\overline{\Delta}_j$ is also paired with a face of this chain of truncated-prisms, then the face $\tilde{\sigma}_i^*$ of $\tilde{\Delta}_i^*$ has an induced pairing with the face $\tilde{\sigma}_j^*$ of $\tilde{\Delta}_j^*$ through the chain of truncated-prisms.

Hence, we get a 3-complex $\tilde{\Delta}^*/\tilde{\Phi}^*$, which is a 3-manifold except, possibly, at its vertices. We will denote the associated ideal triangulation by \mathcal{T}^* . We call \mathcal{T}^* the ideal triangulation obtained by *crushing the triangulation \mathcal{T} along S* . We denote the image of a tetrahedron $\tilde{\Delta}_i^*$ by Δ_i^* and, as above, call $\tilde{\Delta}_i^*$ the lift of Δ_i^* .

We have the following version of the Fundamental Theorem for Crushing Triangulations along a Normal Surface. A more general version and its proof appear in [5].

3.1. Theorem. *Suppose \mathcal{T} is a triangulation of a compact, orientable 3-manifold or an ideal triangulation of the interior of a compact, orientable 3-manifold M . Suppose S is a closed normal surface embedded in M , X is the closure of a component of the complement of S , and X does not contain any vertices of \mathcal{T} . If*

- i) $X \neq \mathbb{P}(\mathcal{C}_X)$
- ii) $\mathbb{P}(\mathcal{C}_X)$ is a trivial product region for X , and
- iii) there are no cycles of truncated prisms in X ,

then the triangulation \mathcal{T} can be crushed along S and the ideal triangulation \mathcal{T}^ obtained by crushing \mathcal{T} along S is an ideal triangulation of $\overset{\circ}{X}$.*

In this situation, we say the triangulation \mathcal{T} admits a *combinatorial crushing along S* . Notice that in the case of a combinatorial crushing, the tetrahedra in the ideal triangulation \mathcal{T}^* are in one-one correspondence with the truncated tetrahedra in the cell decomposition \mathcal{C}_X of X . The latter come from truncating a sub collection of the tetrahedra of \mathcal{T} and can be thought of as actually being a sub collection of the tetrahedra of \mathcal{T} ; the face identifications for \mathcal{T}^* are induced by the face identifications of \mathcal{T} .

3.2. Inflating ideal triangulations. A triangulation \mathcal{T} of the compact 3-manifold M is said to be a *minimal-vertex triangulation* if for any other triangulation \mathcal{T}_1 of M the number of vertices of \mathcal{T} is no more than the number of vertices of \mathcal{T}_1 , $|\mathcal{T}^{(0)}| \leq |\mathcal{T}_1^{(0)}|$. If M is closed, then M has a one-vertex triangulation; hence, a minimal-vertex triangulation of M is a one-vertex triangulation [10]. If M is a compact 3-manifold with boundary, no component of which is a 2-sphere, then M has a triangulation with all of its vertices in the boundary and then just one vertex in each boundary component; hence, for such a manifold a minimal-vertex triangulation has all the vertices in the boundary and then just one vertex in each boundary component [5]. These are the triangulations we are interested in and rather than write all of this out, we just say minimal-vertex triangulation.

If M is a compact 3-manifold with boundary and \mathcal{T} is a triangulation of M with normal boundary, then if \mathcal{T} admits a crushing along the normal boundary, we say \mathcal{T} can be *crushed along ∂M* .

If S is a triangulated surface, we say that a subcomplex ξ in the 1-skeleton of the triangulation of S is a *frame* in S , if ξ is a spine for S (its complement is a connected open cell in S) and is a minimum among spines, with respect to set inclusion. In any triangulation of S there are many choices for a frame. See Figure 2 for examples of frames in a torus. A vertex in a frame is called a *branch* or *branch point* if it has index greater than two. The closure of a component of a frame minus its branch points is called a *branch*. For the examples in Figure 2, that on the left has one branch point of index 4 and two branches while that on the right has two branch points, each of index 3, and three branches.

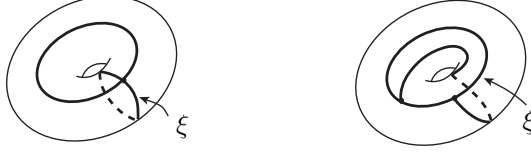


FIGURE 2. There are only two possible topological types for frames in a triangulation of the torus.

3.2. Definition. If \mathcal{T}^* is an ideal triangulation of $\overset{\circ}{X}$, the interior of the compact 3-manifold X , an *inflation* of \mathcal{T}^* is a minimal-vertex triangulation \mathcal{T} of X with normal boundary that admits a combinatorial crushing along ∂X for which the ideal triangulation obtained by crushing \mathcal{T} along ∂X is the ideal triangulation \mathcal{T}^* of $\overset{\circ}{X}$.

A construction for inflations of ideal triangulations of 3-manifolds is developed in [9]. We discuss the construction here but reference the reader to [9] for complete details. For a given ideal triangulation of the interior of a compact 3-manifold, there is not a unique inflation; however, all inflations of a given ideal triangulation share many common properties, some of which play a crucial role in this work.

Our construction begins with the choice of a “frame” in the 1-skeleton of the induced triangulation of each vertex-linking surface of \mathcal{T}^* . If v^* is an ideal vertex of \mathcal{T}^* , we will use the notation S_{v^*} for the vertex-linking surface of v^* and ξ for a frame in S_{v^*} . If there are a number of ideal vertices, then for an ideal vertex v_i^* we use $S_{v_i^*}$ for the vertex-linking surface and ξ_i for a frame in $S_{v_i^*}$. We let $\Lambda = \xi_1 \cup \xi_2 \cup \dots \cup \xi_k$ denote the union of the frames from all the vertex-linking surfaces.

An inflation of an ideal triangulation \mathcal{T}^* of $\overset{\circ}{X}$ includes all the tetrahedra of \mathcal{T}^* and then, guided by the frame Λ , new tetrahedra are added to the tetrahedra of \mathcal{T}^* and new face identifications are determined (discarding some of the face identifications of \mathcal{T}^* , using some of the face identifications of \mathcal{T}^* , and adding some new face identifications) to arrive at a minimal-vertex triangulation \mathcal{T}_Λ of X . The triangulation \mathcal{T}_Λ will have normal boundary that admits a combinatorial crushing of \mathcal{T}_Λ along ∂X , crushing \mathcal{T}_Λ back to the ideal triangulation \mathcal{T}^* . Figure 3 provides a schematic for going between an ideal triangulation \mathcal{T}^* of $\overset{\circ}{X}$ and a minimal-vertex triangulation \mathcal{T}_Λ of X .

An ideal vertex v^* in \mathcal{T}^* inflates to a minimal (one-vertex) triangulation of a component B_v of ∂X , which is induced by \mathcal{T}_Λ . The vertex-linking surface S_{v^*} about the ideal vertex v^* inflates to a normal boundary S_v in the triangulation \mathcal{T}_Λ , which is boundary-linking B_v .

3.3. Closed normal surfaces. In [9] a one-one correspondence is given between the closed normal surfaces in an ideal triangulation \mathcal{T}^* and the closed normal surfaces in any inflation \mathcal{T} of \mathcal{T}^* . A special case of this relationship is a key ingredient for the work in this paper and relates boundary parallel normal surfaces in an inflation of an ideal triangulation with normal surfaces parallel to vertex-linking surfaces in the ideal triangulation. We provide the details for this special case in this section.

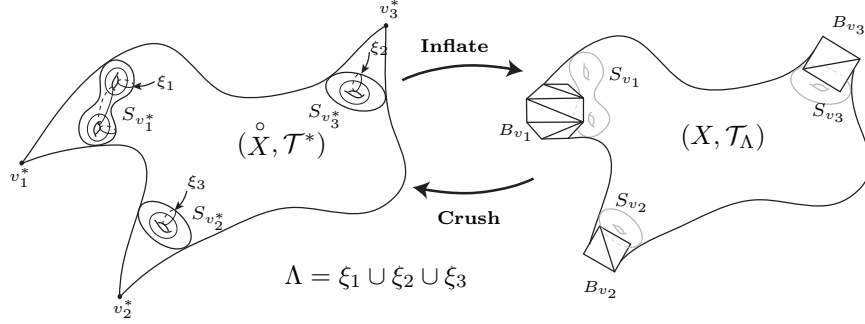


FIGURE 3. Schematic of an inflation of an ideal triangulation using the collection of frames in Λ

3.3. Lemma. *Suppose M is a compact 3-manifold with nonempty boundary and \mathcal{T} is a triangulation of M with normal boundary. An embedded normal surface in \mathcal{T} that contains all the quad types of a boundary-linking surface has that boundary-linking surface as a component.*

Proof. Suppose S is an embedded normal surface in \mathcal{T} , \tilde{B} is a boundary-linking surface, and all quad types of \tilde{B} are represented as quad types in S . Then S and \tilde{B} are contained in the carrier of S , a face of compatible (no two distinct quad types in the same tetrahedron) normal solutions in the solution space of embedded normal surfaces. It may be the case that \tilde{B} is in a proper face. Since, \tilde{B} has no more quad types than S , it follows that there is a normal surface R and positive integers k, n , and m so that $kS = nR + m\tilde{B}$. However, we can move \tilde{B} by a normal isotopy so that it does not meet R . Hence, we have \tilde{B} a component of kS and therefore a component of S . \square

3.4. Lemma. *Suppose M is a compact 3-manifold with nonempty boundary, no component of which is a 2-sphere. Suppose \mathcal{T}^* is an ideal triangulation of $\overset{\circ}{M}$, and \mathcal{T} is an inflation of \mathcal{T}^* . The combinatorial crushing map determined by crushing \mathcal{T} along ∂M takes a closed normal surface S in \mathcal{T} to a closed normal surface S^* in \mathcal{T}^* ; furthermore, S and S^* are homeomorphic.*

Proof. Let X denote the component of the complement of the boundary-linking surfaces that does not meet ∂M . Then X contains none of the vertices of \mathcal{T} and has a nice cell-decomposition \mathcal{C} ; furthermore, this cell-decomposition combinatorially crushes along the boundary-linking surfaces to the ideal triangulation \mathcal{T}^* .

Let S be a closed normal surface in \mathcal{T} . The surface S has an induced cell-decomposition from \mathcal{T} consisting of normal quadrilaterals and normal triangles. Since S is a closed normal surface, we may assume S does not meet any of the boundary-linking surfaces along which we are crushing, and thus $S \subset X$.

If a normal quad or normal triangle of S is in a truncated-tetrahedron in \mathcal{C} , then upon crushing, the truncated-tetrahedron is taken to a tetrahedron of \mathcal{T}^* and the normal cells of S in the truncated-tetrahedron are carried isomorphically onto normal cells in \mathcal{T}^* (see Figure 4 A). If a normal quad or normal triangle of S is in a truncated-prism of \mathcal{C} , then the truncated-prism is crushed to a face in \mathcal{T}^* and the normal cells of S are crushed to normal arcs in that face. The normal

arcs in the hexagonal faces of the truncated-prisms correspond to where S meets these hexagonal faces and are matched under the crushing map from the various truncated prisms in a chain of truncated prisms. Arcs in the trapezoidal faces of the truncated-prism crush to points in the edges of the face in which the truncated prism crushes (see Figure 4 B). Finally, the normal cells of S in the product blocks of \mathcal{C} are “horizontal” triangles in the triangular product blocks and “horizontal” quadrilaterals in the quadrilateral blocks and, hence, each is crushed to a single point in an edge of \mathcal{T}^* (see Figure 4 C). The crushing in the trapezoidal faces of the truncated-prisms and the product blocks are consistent. It follows that the image of S is formed from the collection of normal triangles and normal quadrilaterals of S that are in the truncated-tetrahedron of \mathcal{C} by identifications along their edges and gives a normal surface S^* in \mathcal{T}^* .

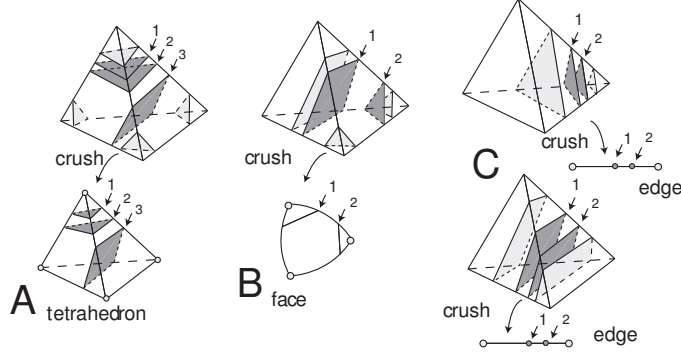


FIGURE 4. A. Normal disks in truncated tetrahedra go to normal disks. B. Normal disks in truncated prisms go to normal arcs. C. Normal disks in product blocks go to points.

To see that S and S^* are homeomorphic, we observe that the inverse image of a point in the interior of a normal quad or normal triangle in S^* is a point in the interior of a normal quad or normal triangle in S . The inverse image of a point in an edge of S^* is either a point in an edge of S or a sequence of arcs in normal quads or normal triangles of S ; there are no cycles of truncated prisms and so no cycles of cells of S in truncated-prisms. The inverse image of a vertex of S^* is a horizontal cross section $K_i \times t$ in one of the component product pieces $\mathbb{P}_i = K_i \times I$ of the combinatorial product $\mathbb{P}(\mathcal{C}_X)$. Hence, K_i is a contractible planar complex. Thus for each point of S^* its inverse image in S is a contractible planar complex and so the combinatorial crushing map gives a cell-like map from S to S^* and by a 2-dimensional versions of [1, 16] it follows that S and S^* are homeomorphic. \square

3.5. Theorem. *Suppose M is a compact 3-manifold with nonempty boundary no component of which is a 2-sphere. Suppose \mathcal{T}^* is an ideal triangulation of $\overset{\circ}{M}$, and \mathcal{T} is an inflation of \mathcal{T}^* . The combinatorial crushing map determined by crushing \mathcal{T} along ∂M induces a bijection between the closed normal surfaces in \mathcal{T} and the closed normal surface in \mathcal{T}^* ; furthermore, corresponding surfaces are homeomorphic.*

Proof. By Lemma 3.4 we only need to show that the combinatorial crushing induces a bijection between the closed normal surfaces of \mathcal{T} and those of \mathcal{T}^* .

First we shall show that the correspondence is injective. Suppose S_1 and S_2 are distinct closed normal surfaces in \mathcal{T} . Since both are closed, we may assume that (up to normal isotopy) they do not meet any boundary-linking surface in \mathcal{T} . Let X denote the component of the complement of the boundary-linking normal surfaces in \mathcal{T} , which does not meet ∂M and let \mathcal{C}_X denote the nice cell decomposition on X induced by \mathcal{T} . Then S_1 and S_2 are distinct normal surfaces in \mathcal{C}_X and hence, have distinct normal coordinates. By Lemma 3.4 the combinatorial crushing of \mathcal{T} to \mathcal{T}^* takes S_1 and S_2 to closed normal surfaces S_1^* and S_2^* in \mathcal{T}^* , respectively. We will show that S_1^* and S_2^* have distinct normal coordinates.

If S_1 and S_2 have distinct sets of normal disks in a truncated tetrahedron of \mathcal{C}_X , then S_1^* and S_2^* have distinct normal disks in a tetrahedron of \mathcal{T}^* and hence, $S_1^* \neq S_2^*$.

If S_1 and S_2 have distinct normal disks in a truncated prism, say π , then they have distinct sets of normal arcs in a hexagonal face of π , which extend to distinct sets of normal arcs on all the hexagonal faces of the truncated prisms in the chain of truncated prisms containing π , which leads to a distinct set of normal disks in a truncated tetrahedron in which the chain terminates. This again gives that $S_1^* \neq S_2^*$. Finally if S_1 and S_2 have a distinct number of quads or triangles in a quadrilateral or triangular block, respectively, then S_1 and S_2 meet an entire product component, $K_i \times I$ in a distinct number of horizontal slices. The vertical frontier of a product $K_i \times I$ is made up of trapezoidal faces which are paired with trapezoidal faces of truncated prisms. Thus we have that S_1 and S_2 must meet a truncated prism in distinct normal disks. From the previous consideration, we have that S_1^* and S_2^* are distinct. So, the correspondence is injective.

Now, we must show the correspondence is surjective. Suppose S^* is a closed normal surface in \mathcal{T}^* . First, we consider how, S^* meets a tetrahedron of \mathcal{T}^* . Each tetrahedron of \mathcal{T}^* is the image of a single truncated tetrahedron of \mathcal{C}_X under the crushing map; hence, there is a unique choice of normal cells in these truncated tetrahedra of \mathcal{C}_X (tetrahedra of \mathcal{T}) mapping to the normal cells of S^* . If α^* is a face of a tetrahedron of \mathcal{T}^* and α^* meets S^* , then the inverse image of α^* is either a single face between two truncated tetrahedra in \mathcal{C}_X or is the image of a chain of truncated prisms in \mathcal{C}_X between two truncated tetrahedra in \mathcal{C}_X . If the inverse image of α^* is a single face matching two truncated tetrahedra, then there are well determined normal cells in each of these truncated tetrahedra determined by the normal cells in S^* . If there is a chain of truncated prisms determined by α^* , then of the three possible families of normal arcs in α^* , only one of the families determines quadrilaterals in any one of the truncated prisms in the chain determined by α . This again determines a unique way to fill in normal disks extending the normal disks in the truncated tetrahedra. Finally, for each product $K_i \times I$ there is a unique number of horizontal slices determined to complete a normal surface S in \mathcal{T} that crushes to S^* . \square

Recall that any ideal triangulation of the interior of a compact 3-manifold M with boundary, no component of which is a 2-sphere, has numerous inflations. By the previous theorem, all of these inflations have isomorphic sets of closed normal surfaces. Here we use isomorphic to mean a bijection between the sets of normal surfaces where corresponding surfaces are homeomorphic.

3.6. Corollary. *Suppose $M \neq \mathbb{B}^3$ is a compact, irreducible and ∂ -irreducible 3-manifold with nonempty boundary. Suppose \mathcal{T}^* is an ideal triangulation of $\overset{\circ}{M}$, and \mathcal{T} is an inflation of \mathcal{T}^* . There is a closed normal surface in \mathcal{T} isotopic into ∂M but not normally isotopic into ∂M if and only if there is a closed normal surface in \mathcal{T}^* that is isotopic into a vertex-linking surface but is not normally isotopic into a vertex-linking surface.*

Proof. Suppose S and S^* are closed normal surfaces in \mathcal{T} and \mathcal{T}^* , respectively, that correspond under the combinatorial crushing map taking \mathcal{T} to \mathcal{T}^* . Then the closure of the components of the complement of S have a correspondence under the combinatorial crushing map and corresponding components are homeomorphic. Also, we have that the correspondence under the combinatorial crushing map takes boundary-linking normal surfaces in \mathcal{T} to vertex-linking normal surfaces in \mathcal{T}^* . Hence, if S is isotopic into ∂M , S is isotopic to a boundary-linking surface and so, S^* is isotopic to a vertex-linking surface in \mathcal{T}^* . The converse also follows. \square

4. ANNULAR-EFFICIENT TRIANGULATIONS

If M is a 3-manifold and \mathcal{T} is a triangulation, we say the triangulation \mathcal{T} is 0-efficient if

- (i) M is closed and the only normal 2-spheres are vertex-linking; or
- (ii) $\partial M \neq \emptyset$ and the only normal disks are vertex-linking.

It is shown in Proposition 5.1 and Proposition 5.15 of [5] that if \mathcal{T} is a 0-efficient triangulation of the compact 3-manifold M , then if

- (i) M is closed, then $M \neq \mathbb{R}P^3$, is irreducible, and \mathcal{T} has only one vertex, or $M = S^3$ and \mathcal{T} has precisely two vertices.
- (ii) $\partial M \neq \emptyset$, then M is irreducible, ∂ -irreducible, there are no normal 2-spheres, all the vertices are in ∂M , and there is precisely one vertex in each boundary component, or $M = \mathbb{B}^3$.

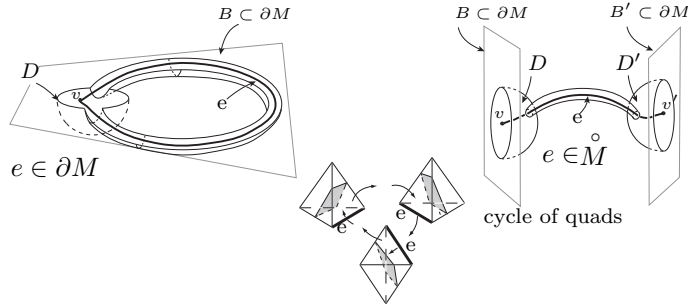


FIGURE 5. Thin edge-linking annuli. On the left is a thin edge-linking annulus about the edge e in the boundary. On the right, the edge e is in the interior of the manifold.

If \mathcal{T} is a triangulation of the 3-manifold M , we say a normal annulus in M is *thin edge-linking* if and only if it is normally isotopic to an arbitrarily small regular neighborhood of an edge in the triangulation. In Figure 5, we give examples of thin edge-linking annuli; the figure on the left is a thin edge-linking annulus for an edge

e in ∂M and the one on the right is a thin edge-linking annulus for an edge e in $\overset{\circ}{M}$. Notice that a thin edge-linking annulus about the edge e is determined from one or two vertex-linking disks by removing all normal triangles that meet e and replacing them with normal quads that do not meet e but are in the tetrahedra containing e . A necessary and sufficient condition for an edge e in ∂M or a properly embedded edge e in M to have a thin edge-linking annulus about it is that no face in the triangulation has two edges identified to e .

If M is a compact 3-manifold with nonempty boundary and \mathcal{T} is a triangulation of M , we say \mathcal{T} is *annular-efficient* if and only if \mathcal{T} is 0-efficient and the only normal, incompressible annuli are thin edge-linking annuli. David Bachman and Saul Schleimer, who have independently studied annular-efficient triangulations, call a triangulation *1/2-efficient* if it is annular-efficient in our sense.

4.1. Proposition. *Suppose $M \neq \mathbb{B}^3$ is a compact 3-manifold with boundary and has an annular-efficient triangulation. Then M is irreducible, ∂ -irreducible, and an-annular. Furthermore, there are no normal 2-spheres, all the vertices are in ∂M , the only normal disks are vertex-linking, and there is precisely one vertex in each component of ∂M .*

Proof. Since an annular-efficient triangulation is 0-efficient, it follows from Theorem 5.15 of [5] that M is irreducible and ∂ -irreducible and there are no normal 2-spheres, all the vertices are in ∂M , and there is precisely one vertex in each boundary component of M . Hence, it remains to prove that M is an-annular. If there is a properly embedded, essential annulus in M , then for any triangulation, there must be a normal, embedded, essential annulus in M . In particular, this would need to be the case for the given annular-efficient triangulation. However, a normal, embedded, essential annulus can not be thin edge-linking, as a thin edge-linking annulus is parallel into the boundary of the manifold and, therefore, is not essential. Thus the assumption of an embedded, essential annulus leads to a contradiction. \square

4.2. Proposition. *Given a triangulation of a compact, orientable 3-manifold with nonempty boundary, no component of which is a 2-sphere, there is an algorithm to decide if the triangulation is annular-efficient. Furthermore, if the triangulation is not 0-efficient, the algorithm will construct a normal disk that is not vertex-linking; and if the triangulation is 0-efficient and is not annular-efficient, the algorithm will construct an incompressible, normal annulus that is not thin edge-linking.*

Proof. By Proposition 5.19 of [5], it can be decided if the given triangulation is 0-efficient; and, if there is a normal disk that is not vertex-linking, the algorithm will construct one. So, we may assume the only normal disks are vertex-linking; i.e., the triangulation is 0-efficient.

If there is an incompressible, normal annulus that is not thin edge-linking, then consider one, say A , where the carrier of A , $\mathcal{C}(A)$, has minimal dimension. If $\mathcal{C}(A)$ is not a vertex, then there are normal surfaces X and Y in proper faces of $\mathcal{C}(A)$ and positive integers k, n and m so that $kA = nX + mY$. Since the triangulation is 0-efficient, the only positive Euler characteristic normal surfaces are vertex-linking normal disks; hence, neither X nor Y has a component with positive Euler characteristic. It follows that the components of both X and Y have Euler characteristic zero. Since A has essential boundary, every component of X and Y with boundary has essential boundary. It follows that the components of X and Y with boundary

are either an annulus or a Möbius band. However, if a component of X or Y is a Möbius band, then we would have a normal annulus that is not thin edge-linking and carried by a proper face of $\mathcal{C}(A)$, which contradicts our choice of A . So, any component of X or Y that has boundary is an annulus and, again by our choice of A , these annuli must be thin edge-linking. Now, a thin edge-linking annulus can be normally isotoped to miss any closed normal surface. It follows that both X and Y must have components with boundary and as such both must have components that are thin edge-linking annuli. However, the Haken sum of two thin edge-linking annuli is either two thin edge-linking annuli (the two annuli are the same or have their boundaries in distinct boundaries of the 3-manifold), or has a component a vertex-linking disk. Both possibilities lead to a contradiction that A is connected and not a thin edge-linking annulus.

It follows that if the triangulation is 0-efficient and there is a normal annulus that is not thin edge-linking, then there is one at a vertex of the projective solution space for the triangulation. Furthermore, we can recognize if a normal surface is a thin edge-linking annulus. \square

The following two results are from [5] and provide converses to Proposition 5.1 and Proposition 5.15 of that work.

4.3. Theorem. *If M is a closed, orientable, irreducible 3-manifold distinct from $\mathbb{R}P^3$, then there is an algorithm that will modify any triangulation of M to a 0-efficient triangulation.*

4.4. Theorem. *If $M \neq \mathbb{B}^3$ is a compact, orientable, irreducible, ∂ -irreducible 3-manifold, with non-empty boundary, then there is an algorithm that will modify any triangulation of M to a 0-efficient triangulation.*

Theorem 4.5 is a converse to Proposition 4.1 and is our main theorem. Bachman and Schleimer communicated to us [2] that they have an independent proof that a compact irreducible, ∂ -irreducible, an-annular 3-manifold admits an annular-efficient triangulation. The proof given here is constructive and follows from the methods introduced in [5]; we believe their methods may be different.

4.5. Theorem. *Suppose $M \neq \mathbb{B}^3$ is a compact, irreducible, ∂ -irreducible, an-annular 3-manifold with nonempty boundary. Then there is an algorithm that will modify any triangulation of M to an annular-efficient triangulation of M .*

Before giving the proof, we outline our approach and provide some results needed in our proof.

(Outline of Proof). We are given a compact 3-manifold M with boundary via a triangulation \mathcal{T} . We are also given that M is irreducible, ∂ -irreducible, and an-annular. Note that given a 3-manifold with boundary, algorithms exist to determine if it is irreducible [15, 8], ∂ -irreducible [3, 8, 14, 17], or an-annular [4, 7]. However, we assume in this work that we are given that the manifold is irreducible, ∂ -irreducible, and an-annular.

If there is a normal disk in \mathcal{T} that is not vertex-linking or a normal annulus with essential boundary in \mathcal{T} that is not thin edge-linking, then by a barrier surface argument, there is a closed normal surface in \mathcal{T} that is isotopic into ∂M but is not normally isotopic into ∂M . Hence, we can prove Theorem 4.5 if we can modify the triangulation \mathcal{T} so that the only normal surface isotopic into a component of ∂M is

boundary-linking (in particular, \mathcal{T} must have a normal boundary). To do this, we first modify the given triangulation to an ideal triangulation of $\overset{\circ}{M}$, the interior of M , and then modify this ideal triangulation to an ideal triangulation of $\overset{\circ}{M}$ having the property that a normal surface isotopic to a vertex-linking surface is also normally isotopic to that vertex-linking surface. We then rebuild a triangulation of M by inflating this ideal triangulation of $\overset{\circ}{M}$ and use Corollary 3.6 to conclude that the inflation is annular-efficient.

Theorems 7.1 and 7.2 of [5] establish, under our hypothesis, the existence of a 0-efficient ideal triangulation of $\overset{\circ}{M}$. Marc Lackenby proved in [13] that with our hypotheses and the additional condition that every boundary component is an annulus, $\overset{\circ}{M}$ admits a taut ideal triangulation, from which it follows that the triangulation also is 0-efficient. While a taut ideal triangulation, which exists under additional hypothesis, implies that the ideal triangulation is 0-efficient, neither the results of [5] or [13] give us what we need for our proof; and in neither of these referenced results do we have constructive proofs. Hence, we first establish a constructive proof that modifies the given triangulation of M to an ideal triangulation of $\overset{\circ}{M}$. Having constructed an ideal triangulation of $\overset{\circ}{M}$, we modify this ideal triangulation, if necessary, to obtain an ideal triangulation in which the only normal surfaces isotopic to a vertex-linking surface are normally isotopic to it. This condition will enable us to construct an annular-efficient triangulation of the compact 3-manifold M via an inflation.

The next theorem requires a more general version of crushing a triangulation than that required earlier for a combinatorial crushing; however, this general version is precisely the version from Section 4 of [5] and our theorem here is a constructive version of Theorem 7.1 of [5].

4.6. Theorem. *Suppose M is a compact, irreducible, ∂ -irreducible, an-annular 3-manifold. Then for any triangulation \mathcal{T} of M , there is an algorithm to modify the triangulation \mathcal{T} to an ideal triangulation of $\overset{\circ}{M}$.*

Proof. M is given by the triangulation \mathcal{T} . Let B_1, \dots, B_n denote the components of ∂M .

If v is a vertex of \mathcal{T} in $\overset{\circ}{M}$, then there is an embedded arc in the 1-skeleton of \mathcal{T} having v as one end point and meeting ∂M only in its other end point, a vertex of \mathcal{T} in ∂M .

Put an order on the vertices of \mathcal{T} in $\overset{\circ}{M}$ and construct a finite number of pairwise disjoint trees, L_1, \dots, L_K in the 1-skeleton of \mathcal{T} so that for each $j, 1 \leq j \leq K$, the tree L_j meets ∂M in a single vertex of \mathcal{T} and every vertex of \mathcal{T} in $\overset{\circ}{M}$ is in L_j for some j . For each $i, 1 \leq i \leq n$, let \hat{B}_i denote the boundary component B_i of M along with all trees L_{i_j} that meet B_i . Let N denote a small regular neighborhood of $\cup_{i=1}^n \hat{B}_i$ and let N_i be the component of N containing $\hat{B}_i, 1 \leq i \leq n$. The frontier of N is a barrier surface for the component of the complement of N that does not meet any \hat{B}_i ; and if E_i is the frontier of the component N_i of N , then E_i is isotopic into the boundary component B_i .

The argument from here follows that in the proof of Theorem 7.1 of [5] except here we want the argument to be constructive. We shall indicate how the steps of

that proof can be made constructive; however, we encourage the reader interested in all the details to look at the presentation in [5].

Shrink each E_i to a stable surface in the component of the complement of the frontier of N not meeting $\cup \hat{B}_i$. This is constructive and we arrive at a pairwise disjoint collection of normal surfaces (and possibly some 2-spheres, interior to tetrahedra, which may be discarded) so that for each E_i , there is precisely one normal surface that is isotopic to E_i and therefore is isotopic into the component B_i of ∂M . We continue to call this, now normal, surface E_i and denote the product region determined by the isotopy of E_i into B_i , by P_i , $1 \leq i \leq n$. There is nothing to verify in this step as the conditions on M , leave no other possibilities.

Let X denote the closure of the component of the complement of $\cup E_i$ that does not meet any P_i ; X is homeomorphic to M . Furthermore, X does not contain any vertices of \mathcal{T} . Hence, we have that the triangulation \mathcal{T} induces a nice cell-decomposition \mathcal{C}_X of X and we can proceed to apply our methods to crush the triangulation \mathcal{T} along the normal surfaces E_1, \dots, E_n .

To do this we must verify that we have the sufficient conditions for crushing a triangulation along a normal surface. In the proof of Theorem 7.1 of [5], we argued that we could assume the collection E_1, \dots, E_n as above satisfied a certain maximal condition. Here we show that we can construct a collection along which we can crush. We may actually discover it before we get to a maximal collection in the sense of [5].

We begin by constructing the combinatorial product $\mathbb{P}(C_X)$. It follows immediately that $\mathbb{P}(C_X) \neq X$; for if $\mathbb{P}(C_X) = X$, then M would be an I-bundle, which contradicts M an-annular.

Next we have to show that we can get to a situation where we have a trivial induced product region.

Recall that a component of the combinatorial product $\mathbb{P}(C_X)$ is a product $K_j \times I$, where $K_j^\varepsilon = K_j \times \varepsilon$, $\varepsilon = 0$ or 1 , and $K_j^0 \subset E_i$ and $K_j^1 \subset E_{i'}$ are isomorphic subcomplexes. In [5] we show that if K_j^ε is not contained in a simply connected region of $E_i(E_{i'})$, then we have a properly embedded 0-weight annulus A_j in $K_j \times I$. However, since X is homeomorphic to M and thus is an-annular, we have $i = i'$ and A_j is isotopic into E_i . $E_i \cup A_j$ along with $\cup_{j \neq i} E_j$ form a barrier and we can construct a new normal surface in place of E_i that is isotopic to E_i but not normally isotopic. This gives us a new collection of normal surfaces along which to consider our conditions for crushing. Since the new surface is not normally isotopic to E_i , it follows from Kneser's Finiteness Theorem [12] that this can only happen a finite number of times. Hence, we eventually have a collection of normal surfaces, again called E_1, \dots, E_n , where E_i is parallel into B_i and the combinatorial product for the component of their complement that does not meet $\cup B_i$, and again denoted $\mathbb{P}(C_X)$, has every component where it meets $\cup E_i$ contained in a simply connected subcomplex of some E_i .

Hence, each component $K_j \times I$ of the combinatorial product $\mathbb{P}(C_X)$ has both its end K_j^0 and K_j^1 contained in simply connected subcomplexes of $\cup E_i$, say D_j^0 and D_j^1 , respectively. While K_j^0 is isomorphic to K_j^1 , it may not be the case that D_j^0 is isomorphic to D_j^1 .

Now, as in [5], we might have that $D_j^0 \subset D_j^1$ (or $D_j^1 \subset D_j^0$). If this is the case and we have $D_j^\varepsilon \subset E_j$, then we can construct, again using barrier surfaces, a new normal surface that is isotopic to E_j but is not normally isotopic to E_j , arriving

at a new collection of normal surfaces, still denoted E_1, \dots, E_n with E_i isotopic to B_i . Again, by Kneser's Finiteness Theorem, this can only happen a finite number of times.

In this way, and after a predicted number of steps, we construct a collection of normal surfaces, E_1, \dots, E_n , where we can fill in any missing pieces in the combinatorial product $\mathbb{P}(\mathcal{C}_X)$ to arrive at a trivial induced product region $\mathbb{P}(X)$ for X . For the very same reasons that $\mathbb{P}(\mathcal{C}_X) \neq X$, we have $\mathbb{P}(X) \neq X$.

This takes care of product blocks in \mathcal{C}_X . We now consider truncated prisms. If there are no cycles of truncated prisms, then we can crush the triangulation along the collection E_1, \dots, E_n constructing the desired ideal triangulation of $\overset{\circ}{M}$. So suppose there is a cycle of truncated prisms. Just as in [5], if the cycle is about a single edge, then there is a surgery on a member of the collection E_1, \dots, E_n , giving a new collection. As before, this can only happen a finite number of times. From the argument in [5] the only possible cycle of truncated prisms about more than one edge would already be included in the induced product region.

Hence, we can crush the triangulation \mathcal{T} along the constructed set of normal surfaces. The crushing gives a set of tetrahedra from the truncated tetrahedra in $\mathcal{C}(X)$ along with face identifications from the original face identifications, possibly translated through a chain of truncated prisms. This gives an ideal triangulation of $\overset{\circ}{M}$. \square

The proof of Theorem 4.6 gives us that any time there is a closed normal surface that is isotopic into ∂M , then under the hypothesis for M we can crush the triangulation along a (possibly different) closed normal surface isotopic into ∂M . It seems that from this we should have a way to show that we eventually arrive at an annular efficient triangulation; however, the problem is that in crushing we arrive at an ideal triangulation of $\overset{\circ}{M}$ and we then need to add tetrahedra to this ideal triangulation to get back to a triangulation of the compact 3-manifold M . We have not been able to show that this approach eventually terminates. So, we switch to getting an ideal triangulation of $\overset{\circ}{M}$ so that the only (closed) normal surface parallel to a vertex-linking surface is the vertex-linking surface. Then we can show any inflation of this ideal triangulation is an annular efficient triangulation of M .

4.7. Theorem. *Suppose M is a compact, irreducible, ∂ -irreducible, an-annular 3-manifold. Then for any ideal triangulation \mathcal{T}' , there is a algorithm to modify the triangulation \mathcal{T}' of $\overset{\circ}{M}$ to an ideal triangulation \mathcal{T}^* of $\overset{\circ}{M}$ having the property that any (closed) normal surface in \mathcal{T}^* that is isotopic to a vertex-linking surface is normally isotopic to that vertex-linking surface.*

Proof. We are given the ideal triangulation \mathcal{T}' of $\overset{\circ}{M}$. By Corollary 2.2 we can decide if there is a closed normal surface in \mathcal{T}' that is isotopic to a vertex-linking surface but is not itself a vertex-linking surface. If there is none, then \mathcal{T}' satisfies the desired conclusion. On the other hand, if there is one, then the algorithm will construct one, say F , and F is isotopic to a vertex-linking surface but is not itself vertex-linking. We wish to crush the triangulation \mathcal{T}' along F . However, to keep the situation consistent with the cell decompositions we like and our methods, if we have ideal vertex-linking surfaces $S_{v_1^*}, \dots, S_{v_n^*}$ and notation has been chosen so

that F is isotopic to $S_{v_1^*}$ but is not normally isotopic to $S_{v_1^*}$, then we wish to crush the triangulation along the collection of surfaces $F, S_{v_2^*}, \dots, S_{v_n^*}$.

If X is the closure of the component of the complement of $F, S_{v_2^*}, \dots, S_{v_n^*}$ not meeting any of the ideal vertices of \mathcal{T}' , then X is homeomorphic to M and we can proceed in finding a collection of normal surfaces along which to crush the triangulation \mathcal{T}' , replacing the collection E_1, \dots, E_n in the proof of Theorem 4.6, by the collection $F, S_{v_2^*}, \dots, S_{v_n^*}$. Hence, we arrive at an ideal triangulation \mathcal{T}^* of $\overset{\circ}{X}$ (homeomorphic with $\overset{\circ}{M}$) obtained by crushing \mathcal{T}' along a collection of normal surfaces with at least one of them not vertex-linking.

The tetrahedra of \mathcal{T}^* come from a subset of the tetrahedra of \mathcal{T}' that become truncated tetrahedra in the cell decomposition \mathcal{C}_X of X . Now, since one of the normal surfaces along which we are crushing is not vertex-linking, it must contain a normal quadrilateral, and hence, at least one of the tetrahedra of \mathcal{T}' gives a truncated prism in \mathcal{C}_X and we have that $|\mathcal{T}^*| < |\mathcal{T}'|$.

It follows that the process must stop and it stops only when we have an ideal triangulation of $\overset{\circ}{M}$ where the only closed normal surface isotopic to a vertex-linking surface is itself a vertex-linking surface. \square

We are now ready to prove Theorem 4.5; we give the statement again for convenience.

Theorem. *Suppose $M \neq \mathbb{B}^3$ is a compact, irreducible, ∂ -irreducible, an-annular 3-manifold with nonempty boundary. Then there is an algorithm that will modify any triangulation of M to an annular-efficient triangulation of M .*

Proof. Suppose \mathcal{T} is a triangulation of M . By Theorem 4.6, there is an algorithm to modify the triangulation \mathcal{T} to an ideal triangulation \mathcal{T}' of $\overset{\circ}{M}$. Now by Theorem 4.7, we can modify the ideal triangulation \mathcal{T}' of $\overset{\circ}{M}$ to an ideal triangulation \mathcal{T}^* of $\overset{\circ}{M}$ so that a (closed) normal surface isotopic to a vertex-linking surface is normally isotopic to that vertex-linking surface. Construct any inflation, say $\mathcal{T}_{\frac{1}{2}}$, of the ideal triangulation \mathcal{T}^* . Then by Corollary 3.6, the triangulation $\mathcal{T}_{\frac{1}{2}}$ of M has the property that a closed normal surface in $\mathcal{T}_{\frac{1}{2}}$ that is isotopic into ∂M is a boundary-linking surface. From our observations above, the triangulation $\mathcal{T}_{\frac{1}{2}}$ can not have a normal disk that is not vertex-linking ($\mathcal{T}_{\frac{1}{2}}$ is 0-efficient) and can not have a normal annulus with essential boundary that is not thin edge-linking ($\mathcal{T}_{\frac{1}{2}}$ is annular-efficient). \square

Notice, for 3-manifolds having connected boundary, then for an annular-efficient triangulation, the only normal annuli are thin edge-linking. Our original attempt was to prove for a manifold M satisfying our hypothesis, then any triangulation of M could be modified to one in which the only normal annuli are thin edge-linking. What we were unable to eliminate is the possibility that the triangulation we have has a normal, compressible annulus that is not thin edge-linking; such an annulus necessarily has boundary which is vertex-linking curves in distinct boundary components of M (a “fat annulus”). We do, however, have the following curious result.

4.8. Proposition. *Suppose \mathcal{T} is a 0-efficient triangulation of the compact 3-manifold $M \neq \mathbb{B}^3$. Then for any edge e of \mathcal{T} having its vertices in distinct boundary components of M , a small regular neighborhood of e is normally isotopic to a thin edge-linking annulus.*

Proof. Recall that for M to have a 0-efficient triangulation, then M is irreducible, ∂ -irreducible, all the vertices of the triangulation are in ∂M , and there is precisely one-vertex in each boundary component. Suppose e is an edge of \mathcal{T} running between distinct boundary components of M . A small regular neighborhood of e is normally isotopic to a thin edge-linking annulus about e if and only if there is no face of \mathcal{T} meeting e in more than one of its edges.

So, suppose there is a face σ of \mathcal{T} meeting e in more than one of its edges. Since e runs between distinct boundary components of M the only possibility is that two edges of σ meet e and σ is a cone. Let e' be the edge of σ forming the base of the cone; i.e., e' is distinct from e .

The edge e' , which bounds a disk, can not be in ∂M as each edge in ∂M is essential in ∂M and M is ∂ -irreducible. If e' is not in ∂M , then e' bounds an embedded disk meeting ∂M at the vertices of e , one of which is also the vertex of e' . Let v be the vertex of e that is not a vertex of e' . Then using that the vertex-link of v is a disk, we can truncate the cone formed by σ and arrive at a disk D' having e' as its boundary and meeting ∂M only at the vertex of e' . Let $N(D')$ be a small regular neighbor of D' . Then the frontier of $N(D')$ consists of a 2-sphere and a properly embedded disk D with boundary a vertex-linking curve in ∂M . The frontier of a small regular neighborhood of e' is a barrier surface and hence, D can be shrunk in the complement of this small neighborhood of e' to a normal disk that is not vertex-linking ($M \neq \mathbb{B}^3$). However, this contradicts that the triangulation is 0-efficient. \square

5. BOUNDARY SLOPES OF SURFACES

If S is a surface and γ is a closed curve in S , then we call the isotopy class of γ a *slope* and refer to it as the slope of γ . It follows from the proof of Proposition 3.2 of [6] that if M is a link-manifold (nonempty boundary and each boundary component is a torus) and M has no essential annuli between distinct boundary components, then for any 0-efficient triangulation \mathcal{T} of M , there are only finitely many boundary slopes for normal surfaces of a bounded Euler characteristic. Hence, for such an M there are only finitely many boundary slopes for incompressible and ∂ -incompressible surfaces of bounded Euler characteristic. We generalize this to general compact, irreducible, ∂ -irreducible, and an-annular 3-manifolds.

First, we have the following lemma.

5.1. Lemma. *Suppose M is a compact 3-manifold with nonempty boundary and \mathcal{T} is a triangulation of M . Furthermore, suppose F' is a normal surface and A is a thin edge-linking annulus about an edge in ∂M . If the Haken sum $F' + A$ is defined, then $F' + A$ is either*

- (i) *The disjoint union $F' \cup A$,*
- (ii) *A normal surface F and a vertex-linking surface, or*
- (iii) *A normal surface F isotopic to F' .*

Proof. Following a small isotopy of A , we have that $F' \cap A$ is at most a finite number of normal spanning arcs running through the normal quads of A (hence, only meeting normal triangles of F'). If $F' \cap A = \emptyset$, then we have conclusion (i). So, assume $F' \cap A \neq \emptyset$.

An arc $\alpha \subset F' \cap A$ cuts off a small disk d_α in F where $\partial d = \alpha \cup \beta$ with $\beta \subset \partial M$. Following a regular exchange along α , a copy of d_α is joined with A forming a

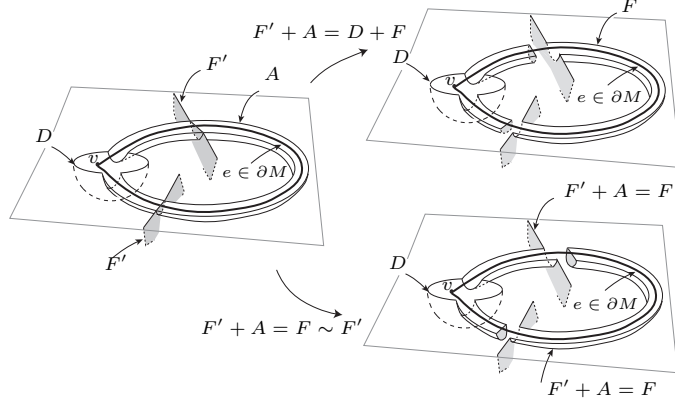


FIGURE 6. Haken sum of normal surface with thin edge-linking annulus.

∂ -compression of A . See Figure 6. Since $F' + A$ is a normal surface, it is only possible that two such ∂ -compressions occur adjacent on A when together they cut off the vertex-linking disk. See the top right-hand drawing in Figure 6. This gives possibility (ii) of our conclusion. Otherwise, we get possibility (iii). See bottom right-hand drawing in Figure 6. \square

5.2. Theorem. *Suppose \mathcal{T} is an annular efficient triangulation of the compact 3-manifold M . Then there are only finitely many boundary slopes for connected normal surfaces in \mathcal{T} of bounded Euler characteristic.*

Proof. A triangulation \mathcal{T} determines a collection of normal surfaces. Among these is a unique collection of normal surfaces (the fundamental normal surfaces) $F_1, \dots, F_K, T_1, \dots, T_M, A_1, \dots, A_N$ such that any normal surface F can be written as a Haken sum $F = \sum_1^K p_k F_k + \sum_1^M q_m T_m + \sum_1^N r_n A_n$, where $p_k, 1 \leq k \leq K; q_m, 1 \leq m \leq M$; and $r_n, 1 \leq n \leq N$ are nonnegative integers, $\chi(F_k) < 0$, T_m a torus or Klein bottle, and A_n an annulus. Since \mathcal{T} is annular-efficient, M is an-annular and, hence, no A_m can be a Möbius band. If we set $F'' = \sum_1^K p_k F_k$, then we have $\chi(F) = \chi(F'')$ and observe for surfaces F with bounded Euler characteristic, there can be only finitely many sums $F'' = \sum_1^K p_k F_k$ of bounded Euler characteristic.

Hence, for bounded Euler characteristic, there are at most a finite number of boundary slopes for surfaces of the form $F' = \sum_1^K p_k F_k + \sum_1^M q_m T_m$. However, for any connected surface $F = F' + A_n$, only one of the possibilities in Lemma 5.1 can hold and that is (iii). It follows that for F connected and $F = F' + \sum_1^N r_n A_n$, we $F \sim F'$ and so F and F' have the same boundary slopes. This proves our theorem. \square

The following corollary is immediate as an incompressible and ∂ -incompressible surface in an irreducible and ∂ -irreducible 3-manifold is isotopic to a normal surface in any triangulation.

5.3. Corollary. *Suppose $M \neq \mathbb{B}^3$ is a compact, irreducible, ∂ -irreducible, annular 3-manifold. Then there are only finitely many boundary slopes for connected, incompressible, and ∂ -incompressible surfaces in M of bounded Euler characteristic*

REFERENCES

- [1] S. Armentrout. Cellular decompositions of 3-manifolds that yield 3-manifolds. *Memoirs Amer. Math. Soc.*, 107:72pp, 1971.
- [2] D. Bachman and S. Schleimer.
- [3] Wolfgang Haken. *Theorie der Normalflächen: Ein Isotopiekriterium für der Kreisknoten.* *Acta Math.*, 105:245–375, 1961.
- [4] Wolfgang Haken. *Über das Homöomorphieproblem der 3-Mannigfaltigkeiten I.* *Math. Zeitschr.*, 80:89–120, 1962.
- [5] W. Jaco and J. H. Rubinstein. 0-efficient triangulations of 3-manifolds. *J. Diff. Geom.*, 65:61–168, 2003. arXiv:math/0207158.
- [6] W. Jaco, J. H. Rubinstein, and E. Sedgwick. Finding planar surfaces in knot- and link-manifolds. *J. Knot Theory and its Appl.*, 18(3):397–446, 2009. arXiv:math/0608700.
- [7] W. Jaco and E. Sedgwick. Decision problems in the space of Dehn fillings. *Topology*, 42:845–906, 2003. ArXiv:math.GT/9811031.
- [8] W. Jaco and J. Tollefson. Algorithms for the complete decomposition of a closed 3-manifold. *Illinois J. Math.*, 39(3):358–406, 1995.
- [9] William Jaco and J. H. Rubinstein. Inflatons of ideal triangulations of 3-manifolds. (in preparation).
- [10] William Jaco and J. Hyam Rubinstein. Layered triangulations of three-manifolds. arXiv:math/0603601.
- [11] E. Kang and J.H. Rubinstein. Ideal triangulations of 3-manifolds II, Taut and angle structures. *Algebraic and Geometric Topology*, 5:1505–1533, 2005. arXiv:math/0410541.
- [12] H. Kneser. *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten.* *Jahresbericht der Deut. Math. Verein.*, 38:248–260, 1929.
- [13] Marc Lackenby. Taut ideal triangulations of 3-manifolds. *Geom. Topol.*, 4:369–395, 2000. arXiv:math/0003132.
- [14] J. H. Rubinstein. Polyhedral minimal surfaces, Heegaard splittings and decision problems for 3-dimensional manifolds. In *Geometric Topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 1–20. Amer. Math. Soc., Providence, RI, 1997.
- [15] H. Schubert. *Bestimmung der Primfaktorzerlegung von Verkettungen.* *Math. Zeitschr.*, 76:116–148, 1961.
- [16] L. Siebenmann. Approximating cellular maps with homeomorphisms. *Topology*, 11:271–294, 1972.
- [17] A. Thompson. Thin position and the recognition problem for S^3 . *Math. Res. Lett.*, 1(5):613–630, 1994.

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078
E-mail address: `jaco@math.okstate.edu`

DEPARTMENT OF MATHEMATICSS AND STATISTICS, UNIVERSITY OF MELBOURNE, PARKVILLE, VIC 3010, AUSTRALIA
E-mail address: `rubin@ms.unimelb.edu.au`